

# Gauge transformations on locally trivial quantum principal fibre bundles

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## Abstract

We consider in this paper gauge transformations on locally trivial quantum principal fibre bundles (QPFB). If  $\mathcal{P}$ ,  $B$ ,  $H$ , are the algebras of the total space, the base space, the structure group of the bundle, left (right) gauge transformations are defined as automorphisms of the left (right)  $B$ -module  $\mathcal{P}$  which are adapted to the coaction of the Hopf algebra  $H$  and to the covering related to the local trivializations. Connections on the QPFB are in general not transformed into connections. For covariant connections, there are analogues of the classical formulas relating the connection and its gauge transform.

This paper is a follow-up of [3]–[5]. We freely use the results of these papers. In [2], gauge transformations are defined as “vertical automorphisms” of the bundle. We start here with the more general definition of [1] (given there only for trivial bundles) and [7] of gauge transformations as convolution invertible linear maps from the Hopf algebra  $H$  to the total space algebra  $\mathcal{P}$  (taken now in a different context) and consider the action of such transformations on differential geometric objects (connections and covariant derivatives).

We define left (right) gauge transformations on a locally trivial QPFB in the sense of [2] and [4] as automorphisms of the left (right)  $B$ -module  $\mathcal{P}$  which are compatible with the right coaction on the total space and with the local trivializations. On a trivial bundle, such gauge transformations correspond to convolution invertible (left case) and twisted convolution invertible (right case) linear maps from  $H$  to  $B$ . On a locally trivial QPFB, the corresponding objects are linear maps from  $H$  to  $\mathcal{P}$  with these properties. If the structure group is a compact quantum group, gauge transformations can be characterized locally by elements of the algebras defined locally by the covering of the basis. Gauge transformations can also be defined on locally trivial quantum vector bundles (QVB) as module automorphism respecting the local trivializations. If a QVB is associated to some QPFB, every gauge transformation of the QPFB induces a gauge transformation on the QVB.

Every gauge transformation on a QPFB induces a module isomorphism of the module of horizontal forms belonging to a differential structure of  $\mathcal{P}$  (the module structure being with respect to the maximal embeddable LC differential algebra related to the differential structure). It follows that the set of covariant derivatives is invariant under gauge transformations. For the set of connections, this is true at least in the following two cases: If the differential structure

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on  $\mathcal{P}$  is defined using the universal calculus on  $H$ , and if the differential structure on  $\mathcal{P}$  uses a bicovariant calculus on  $H$ , restricting at the same time to “vertical automorphisms”. In any case, the local connection forms and the curvature forms are transformed in the typical (inhomogeneous respectively homogeneous) way. We also give an example of a gauge transformation on a  $U(1)$  bundle over a gluing of  $SU_\nu(2)$  with some “quantum cylinder”, which is not an algebra automorphism.

## 1 Gauge transformations

In [2] gauge transformation on a QPFB  $\mathcal{P}$  are defined as automorphisms  $\alpha : \mathcal{P} \longrightarrow \mathcal{P}$  fulfilling the conditions

$$(\alpha \otimes id) \circ \Delta_{\mathcal{P}} = \Delta_{\mathcal{P}} \circ \alpha \quad (1)$$

$$\alpha \circ \iota = \iota. \quad (2)$$

On trivial QPFB gauge transformations are in one to one correspondence with homomorphisms  $\tau_\alpha : H \longrightarrow B$  fulfilling

$$\begin{aligned} \tau_\alpha(1) &= 1 \\ \tau_\alpha(h)a &= a\tau_\alpha(h), \quad \forall h \in H \quad \forall a \in B \end{aligned}$$

such that

$$\alpha(a \otimes h) = \sum a\tau_\alpha(h_1) \otimes h_2.$$

To get nonclassical gauge transformations one needs a more general definition. First, let us define gauge transformation on trivial QPFB.

**Definition 1** *Let  $B \otimes H$  be a trivial QPFB. A left (right) gauge transformation on  $B \otimes H$  is a left (right)  $(B \otimes 1)$ -module isomorphism  $\alpha : B \otimes H \longrightarrow B \otimes H$  satisfying*

$$(\alpha \otimes id) \circ (id \otimes \Delta) = (id \otimes \Delta) \circ \alpha \quad (3)$$

$$\alpha \circ (id \otimes 1) = (id \otimes 1). \quad (4)$$

Remark: Obviously, for every gauge transformation  $\alpha$ , the inverse  $\alpha^{-1}$  is also a gauge transformation.

In the sequel, our standard notation will be  $\alpha_l$  for left and  $\alpha_r$  for right gauge transformations. The notation  $\alpha_{l,r}$  will be used if something is true for both left and right gauge transformations.

**Proposition 1** *Left (right) gauge transformations  $\alpha_{l,r}$  on a trivial bundle  $B \otimes H$  are in one to one correspondence to linear maps  $\tau_{\alpha_{l,r}} : H \longrightarrow B$  satisfying*

$$\begin{aligned} \tau_{\alpha_{l,r}}(1) &= 1 \\ \sum \tau_{\alpha_l^{-1}}(h_1)\tau_{\alpha_l}(h_2) &= \sum \tau_{\alpha_l}(h_1)\tau_{\alpha_l^{-1}}(h_2) = \varepsilon(h)1 \\ \sum \tau_{\alpha_r^{-1}}(h_2)\tau_{\alpha_r}(h_1) &= \sum \tau_{\alpha_r}(h_2)\tau_{\alpha_r^{-1}}(h_1) = \varepsilon(h)1 \end{aligned}$$

such that

$$\alpha_l(a \otimes h) = \sum a\tau_{\alpha_l}(h_1) \otimes h_2 \quad (5)$$

$$\alpha_r(a \otimes h) = \sum \tau_{\alpha_r}(h_1)a \otimes h_2. \quad (6)$$

Proof: We define

$$\tau_{\alpha_{l,r}}(h) = (id \otimes \varepsilon) \circ \alpha_{l,r}(1 \otimes h). \quad (7)$$

Because of formula (3) and  $(id \otimes \varepsilon \otimes id) \circ (id \otimes \Delta) = id$  we obtain for left gauge transformations

$$\begin{aligned} \alpha_l(a \otimes h) &= (id \otimes \varepsilon \otimes id) \circ (id \otimes \Delta) \circ \alpha_l(a \otimes h) \\ &= (a \otimes 1)(id \otimes \varepsilon \otimes id) \circ (\alpha_l \otimes id) \circ (id \otimes \Delta)(1 \otimes h) \\ &= (a \otimes 1) \sum (\tau_{\alpha_l}(h_1) \otimes h_2) \\ &= \sum (a \tau_{\alpha_l}(h_1) \otimes h_2) \end{aligned}$$

and for right gauge transformations

$$\begin{aligned} \alpha_r(a \otimes h) &= (id \otimes \varepsilon \otimes id) \circ (id \otimes \Delta) \circ \alpha_r(a \otimes h) \\ &= (id \otimes \varepsilon \otimes id) \circ (\alpha_r \otimes id) \circ (id \otimes \Delta)(1 \otimes h)(a \otimes 1) \\ &= \sum (\tau_{\alpha_r}(h_1) \otimes h_2)(a \otimes 1) \\ &= \sum (\tau_{\alpha_r}(h_1) a \otimes h_2). \end{aligned}$$

Since the gauge transformations are left (right)  $(B \otimes 1)$ -module isomorphisms, the properties claimed for  $\tau_{\alpha_{l,r}}$  are easily verified. We leave the other direction of the proof to the reader.  $\square$

**Definition 2** Let  $\mathcal{P}$  be a locally trivial QPFB with local trivializations  $\chi_i : \mathcal{P} \longrightarrow B_i \otimes H$ . A left (right) gauge transformation on  $\mathcal{P}$  is a left  $\iota(B)$ -module isomorphism  $\alpha_{l,r} : \mathcal{P} \longrightarrow \mathcal{P}$  such that there exists a family  $(\alpha_{l,r_i})_{i \in I}$  of left (right) gauge transformation on the trivializations  $B_i \otimes H$  satisfying

$$\chi_i \circ \alpha_{l,r} = \alpha_{l,r_i} \circ \chi_i. \quad (8)$$

**Proposition 2** A left (right) gauge transformation fulfills

$$\Delta_{\mathcal{P}} \circ \alpha_{l,r} = (\alpha_{l,r} \otimes id) \circ \Delta_{\mathcal{P}}. \quad (9)$$

Proof: Using the definitions one calculates

$$\begin{aligned} (\chi_i \circ \alpha_{l,r} \otimes id) \circ \Delta_{\mathcal{P}} &= (\alpha_{l,r_i} \circ \chi_i \otimes id) \circ \Delta_{\mathcal{P}} \\ &= ((\alpha_{l,r_i} \otimes id) \circ (id \otimes \Delta)) \circ \chi_i \\ &= (id \otimes \Delta) \circ \alpha_{l,r_i} \circ \chi_i \\ &= (id \otimes \Delta) \circ \chi_i \circ \alpha_{l,r} \\ &= (\chi_i \otimes id) \circ \Delta_{\mathcal{P}} \circ \alpha_{l,r}. \end{aligned}$$

(9) follows from  $\bigcap_i \ker \chi_i = 0$ .  $\square$

Remark: Left (right) gauge transformations  $\alpha_{l,r}$  can be equivalently defined as left (right)  $\iota(B)$  module isomorphisms  $\alpha_{l,r} : \mathcal{P} \longrightarrow \mathcal{P}$  fulfilling

$$\begin{aligned} \Delta_{\mathcal{P}} \circ \alpha_{l,r} &= (\alpha_{l,r} \otimes id) \circ \Delta_{\mathcal{P}} \\ \alpha_{l,r} \circ \iota &= \iota \\ \alpha(\ker \chi_i) &= \ker \chi_i; \quad \forall i \in I. \end{aligned}$$

**Proposition 3** *The set  $G_{l,r}$  of all left (right) transformations is a group with the composition of maps as group multiplication.*  $\square$

**Proposition 4** *Left (right) gauge transformations  $\alpha_{l,r}$  on a locally trivial QPFB  $\mathcal{P}$  are in one to one correspondence to linear maps  $g_{\alpha_{l,r}} : H \longrightarrow \mathcal{P}$  satisfying*

$$g_{\alpha_{l,r}}(1) = 1 \quad (10)$$

$$\Delta_{\mathcal{P}}(g_{\alpha_l}(h)) = \sum g_{\alpha_l}(h_2) \otimes S(h_1)h_3 \quad (11)$$

$$\Delta_{\mathcal{P}}(g_{\alpha_r}(h)) = \sum g_{\alpha_r}(h_2) \otimes h_3 S^{-1}(h_1) \quad (12)$$

$$\sum g_{\alpha_l}(h_1)g_{\alpha_l^{-1}}(h_2) = \sum g_{\alpha_l^{-1}}(h_1)g_{\alpha_l}(h_2) = \varepsilon(h)1 \quad (13)$$

$$\sum g_{\alpha_r}(h_2)g_{\alpha_r^{-1}}(h_1) = \sum g_{\alpha_r^{-1}}(h_2)g_{\alpha_r}(h_1) = \varepsilon(h)1. \quad (14)$$

The correspondence is given by

$$\alpha_l(f) = \sum f_0 g_{\alpha_l}(f_1) \quad (15)$$

$$\alpha_r(f) = \sum g_{\alpha_r}(f_1) f_0. \quad (16)$$

Proof: We will give the proof only for left gauge transformations because it works for right gauge transformations with the same arguments.

Assume that there is given a linear map  $g : H \longrightarrow \mathcal{P}$  with the properties  $\Delta_{\mathcal{P}}(g(h)) = \sum g(h_2) \otimes S(h_1)h_2$  and  $g(1) = 1$  such that there exists a linear map  $g^{-1} : H \longrightarrow \mathcal{P}$  fullfilling  $\sum g^{-1}(h_1)g(h_2) = \sum g(h_1)g^{-1}(h_2) = \varepsilon(h)1$ . It is easy to verify that the linear map  $\alpha : \mathcal{P} \longrightarrow \mathcal{P}$  defined by

$$\alpha(f) := \sum f_0 g(f_1)$$

is a left gauge transformation.

The proof of the other direction is more complicated. By definition for a given left gauge transformation  $\alpha_l$  there exists a family  $(\alpha_{l_i})_{i \in I}$  of left gauge transformations on the trivializations  $B_i \otimes H$ . Since the linear maps  $\alpha_{l_i}$  are left  $(B_i \otimes 1)$ -module isomorphisms, there exist left  $(B_{ij} \otimes 1)$  module isomorphisms  $\alpha_{l_i}^j : B_{ij} \otimes H \longrightarrow B_{ij} \otimes H$  satisfying

$$\alpha_{l_i}^j \circ (\pi_j^i \otimes id) = (\pi_j^i \otimes id) \circ \alpha_{l_i}. \quad (17)$$

These  $\alpha_{l_i}^j$  satisfy the identity

$$\alpha_{l_i}^j = \phi_{ij} \circ \alpha_{j_i}^i \circ \phi_{ji}, \quad (18)$$

where  $\phi_{ij}$  are the isomorphisms  $\phi_{ij} : B_{ij} \otimes H \longrightarrow B_{ij} \otimes H$  induced from the transition functions  $\tau_{ij}$  of the bundle  $\mathcal{P}$  (see [5]). (18) is proved as follows. Let  $f \in \mathcal{P}$ . We know,

$$(\pi_j^i \otimes id) \circ \chi_i(f) = \phi_{ij} \circ (\pi_i^j \otimes id) \circ \chi_j(f), \quad (19)$$

therefore

$$(\pi_j^i \otimes id) \circ \chi_i(\alpha_l(f)) = \phi_{ij} \circ (\pi_i^j \otimes id) \circ \chi_j(\alpha_l(f)).$$

With (8) and (17) follow the equations

$$(\pi_j^i \otimes id) \circ \alpha_{l_i} \circ \chi_i(f) = \phi_{ij} \circ (\pi_i^j \otimes id) \circ \alpha_{j_i} \circ \chi_j(f) \quad (20)$$

$$\alpha_{l_i}^j \circ (\pi_j^i \otimes id) \circ \chi_i(f) = \phi_{ij} \circ \alpha_{j_i}^i \circ (\pi_i^j \otimes id) \circ \chi_j(f). \quad (21)$$

Inserting (19) in (21) one obtains

$$\alpha_{l_i}^j \circ \phi_{ij} = \phi_{ij} \circ \alpha_{j_l}^i,$$

which proves (18). Because of this formula, the linear maps  $\tau_{\alpha_{l_i}} : H \longrightarrow B_i$  corresponding to the  $\alpha_{l_i}$  satisfy

$$\pi_j^i(\tau_{\alpha_{l_i}}(h)) = \sum \tau_{ij}(h_1) \pi_i^j(\tau_{\alpha_{j_l}}(h_2)) \tau_{ji}(h_3). \quad (22)$$

Now we define a family of linear maps  $g_{\alpha_{l_i}} : H \longrightarrow B_i \otimes H$  by

$$g_{\alpha_{l_i}}(h) := \sum \tau_{\alpha_{l_i}}(h_2) \otimes S(h_1)h_3.$$

It is easy to see that

$$\alpha_{l_i}(a \otimes h) = \sum (a \otimes h_1) g_{\alpha_{l_i}}(h_2).$$

Because of formula (22) the family of linear maps  $(g_{\alpha_{l_i}})_{i \in I}$  fulfills

$$(\pi_j^i \otimes id)(g_{\alpha_{l_i}}(h)) = \phi_{ij} \circ (\pi_i^j \otimes id)(g_{\alpha_{j_l}}(h)), \quad (23)$$

i.e. there exists a unique linear map  $g_{\alpha_l} : H \longrightarrow \mathcal{P}$  satisfying  $\chi_i(g_{\alpha_l}(h)) = g_{\alpha_{l_i}}(h)$  and  $\alpha_l(f) = \sum f_0 g_{\alpha_l}(f_1)$ . The properties of  $g_{\alpha_l}$  are now easily verified by using the properties of the family  $(g_{\alpha_{l_i}})_{i \in I}$ .  $\square$

**Proposition 5** *There is a bijection between left and right gauge transformations.*

Proof: Left and right gauge transformations are related by  $g_{\alpha_r} := g_{\alpha_l} \circ S^{-1}$ .  $\square$

**Proposition 6** *Let  $\Gamma(\mathcal{P})$  be a differential structure on  $\mathcal{P}$  and let  $\alpha_{l,r}$  be a left (right) gauge transformation on  $\mathcal{P}$ .*

*Then the formulas*

$$\alpha_l(\gamma) := \sum \gamma_0 g_{\alpha_l}(\gamma_1) \quad (24)$$

$$\alpha_r(\gamma) := \sum g_{\alpha_r}(\gamma_1) \gamma_0 \quad (25)$$

*define left (right)  $\Gamma_m(B)$ -module isomorphisms  $\alpha_{l,r} : \text{hor}\Gamma_c(\mathcal{P}) \longrightarrow \text{hor}\Gamma_c(\mathcal{P})$ .*

( $\Gamma_m(B)$  is the maximal embeddable LC-differential algebra induced from  $\Gamma_c(\mathcal{P})$ , see [5].) We leave the proof to the reader.

The concept of gauge transformations can be carried over to associated vector bundles. In general one can define gauge transformations on a locally trivial QVB  $E$  as follows:

**Definition 3** *Let  $((E, B, \kappa), V, (\zeta_i, J_i)_{i \in I})$  be a locally trivial QVB. A gauge transformation on  $E$  is a automorphism  $\eta : E \longrightarrow E$  with the properties*

$$\kappa(a) \circ \eta = \eta \circ \kappa(a), \quad \forall a \in B \quad (26)$$

$$\eta(\ker \zeta_i) = \ker \zeta_i. \quad (27)$$

The last condition of this definition has the consequence that there are gauge transformations  $\eta_i : B_i \otimes V \longrightarrow B_i \otimes V$  and  $\eta_{ij}^i : B_{ij} \otimes V \longrightarrow B_{ij} \otimes V$  such that

$$\begin{aligned} \eta_i \circ \zeta_i &= \zeta_i \circ \eta \\ \eta_{ij}^i \circ (\pi_j^i \otimes id) &= (\pi_j^i \otimes id) \circ \eta_i. \end{aligned}$$

**Proposition 7** *Gauge transformations on a locally trivial QVB  $E$  are in one-to-one correspondence with families of gauge transformations  $\eta_i : B_i \otimes V \longrightarrow B_i \otimes V$  satisfying*

$$\eta_{ij}^i = \phi_{ij_E} \circ \eta_{ij}^j \circ \phi_{ji_E}. \quad (28)$$

We omit the proof because it is quite analogous to the proof of Proposition 4 of [5].

**Proposition 8** *Let  $E(\mathcal{P}, F)$  be an associated vector bundle. Every left gauge transformation on  $\mathcal{P}$  determines a gauge transformation on  $E(\mathcal{P}, F)$ .*

Proof: Let  $\alpha_l$  be a left gauge transformation on  $\mathcal{P}$ . The linear map  $\eta_{\alpha_l} : E(\mathcal{P}, F) \longrightarrow E(\mathcal{P}, F)$  defined by

$$\eta_{\alpha_l} := \alpha_l \otimes id$$

is seen to be a gauge transformation on  $E(\mathcal{P}, F)$ .  $\square$

Let  $\Gamma_m(B)$  be the maximal embeddable LC-differential algebra over  $B$  induced from the differential structure on  $\mathcal{P}$ , and let  $E_\Gamma(\mathcal{P}, F)$  be the locally trivial QVB constructed in terms of  $E(\mathcal{P}, F)$  and  $\Gamma_m(B)$ . Because of Proposition 6, one can extend every gauge transformation on  $E(\mathcal{P}, F)$  determined by a gauge transformation on  $\mathcal{P}$  to a module automorphism  $\eta_\alpha$  of  $E_\Gamma(\mathcal{P}, F)$  by

$$\eta_{\alpha_l} := \epsilon_\Gamma \circ (\alpha_l \otimes id) \circ \epsilon_\Gamma^{-1}.$$

We end up this section with some remarks about the general structure of gauge transformations in the case when the structure group of the locally trivial QPFB is a compact quantum group. For the algebra  $P(G)$  of polynomial functions over such a compact quantum group  $G$  one can construct the following linear basis ( see [9], [8] and [6]). Let  $M$  be the set of all irreducible unitary matrix co-representations of  $G$ .

(A unitary matrix co-representation is defined by an  $P(G)$ -valued  $N \times N$ -matrix  $(u_{ij})_{i,j=1,2,\dots,N}$  with  $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kl}$  and  $u_{ij}^* = S(u_{ji})$ .)

Two co-representations  $\rho$  and  $\sigma$  are equivalent if there exists an intertwining operator. This defines an equivalence relation  $\sim$  in  $M$ . Now one can select in every class  $\alpha = M / \sim$  a matrix co-representation  $(u_{ij}^\alpha)_{i,j=1,2,\dots,N^\alpha}$ . It is proved in [9], [8] and [6] that the set of elements  $(u_{ij}^\alpha)_{\alpha \in M/\sim; i,j=1,2,\dots,N^\alpha}$  is a linear basis of  $P(G)$ . This leads us to the following conclusion.

**Proposition 9** *Let  $\mathcal{P}$  be a locally trivial QPFB where the structure group is a compact quantum group. The set of left (right) gauge transformations is in one to one correspondence with sets of invertible  $B_i$ -valued matrices  $(b_{ikl}^\alpha)_{k,l=1,2,\dots,N^\alpha}$   $\alpha \in M/\sim$  satisfying*

$$\pi_j^i(b_{ikl}^\alpha) = \sum_{m,n}^{N^\alpha} \tau_{ij}(u_{km}^\alpha) \pi_i^j(b_{mnn}^\alpha) \tau_{ji}(u_{ml}^\alpha). \quad (29)$$

More precisely (see also formula (22)), one can construct a gauge transformation by mapping every matrix  $u_{ij}^\alpha$  to a set of invertible  $B_i$ -valued  $N^\alpha \times N^\alpha$ -matrices satisfying (29). Doing this for each  $\alpha$  one obtains linear maps  $\tau_i : P(G) \longrightarrow B_i$  defined by

$$\tau_i(u_{ikl}^\alpha) = b_{ikl}^\alpha.$$

Since the matrices  $(b_{ikl}^\alpha)$  are invertible the  $\tau_i$  are convolution invertible. Since formula (29) is fulfilled for each  $\alpha$  the  $\tau_i$  satisfy also (22) and it follows that they determine a gauge transformation.

## 2 Gauge transformations and connections

Since we have extended gauge transformations only to the subalgebra of horizontal forms  $hor\Gamma_c(\mathcal{P})$  it is not possible to transform a connection by a gauge transformation analogous to the classical case by transforming the connection form  $\omega_{l,r}$ . What we can transform is the covariant derivation  $D_{l,r} := hor \circ d$  corresponding to the connection.

Let  $D_{l,r}$  be the left (right) covariant derivative corresponding to a left (right) connection and let  $\alpha_{l,r}$  be a left (right) gauge transformation. One defines the linear map  $D'_{l,r} : hor\Gamma_c(\mathcal{P}) \rightarrow hor\Gamma_c(\mathcal{P})$  by

$$D'_{l,r} := \alpha_{l,r} \circ D_{l,r} \circ \alpha_{l,r}^{-1}.$$

In the sequel we want to discuss this formula for connections on trivial QPFB. We know, that a left (right) connection corresponds to a linear map  $A_{l,r} : H \rightarrow \Gamma^1(B)$  satisfying (54), (55), (79) and (80) of [4] respectively. We are interested in the transformed maps  $A'_{l,r}$  belonging to the  $D'_{l,r}$ . One calculates

$$\begin{aligned} D'_l(a \otimes h) &= \alpha_l D_l \circ \alpha_l^{-1}(a \otimes h) \\ &= \alpha_l \circ hor_l \circ d \sum (a \tau_{\alpha_l^{-1}}(h_1) \otimes h_2) \\ &= \alpha_l \circ hor_l \left( \sum ((da) \tau_{\alpha_l^{-1}}(h_1)) \hat{\otimes} h_2 + \sum a \tau_{\alpha_l^{-1}}(h_1) \hat{\otimes} dh_2 \right) \\ &= \alpha_l \left( \sum (da \tau_{\alpha_l^{-1}}(h_1)) \hat{\otimes} h_2 - \sum a \tau_{\alpha_l^{-1}}(h_1) A_l(h_2) \hat{\otimes} h_3 \right) \\ &= \sum ((da) \tau_{\alpha_l^{-1}}(h_1)) \tau_{\alpha_l}(h_2) \hat{\otimes} h_3 - \sum a \tau_{\alpha_l^{-1}}(h_1) A_l(h_2) \tau_{\alpha_l}(h_3) \hat{\otimes} h_4 \\ &= (da) \hat{\otimes} h - \sum a \tau_{\alpha_l^{-1}}(h_1) d \tau_{\alpha_l}(h_2) \hat{\otimes} h_3 - \sum a \tau_{\alpha_l^{-1}}(h_1) A_l(h_2) \tau_{\alpha_l}(h_3) \hat{\otimes} h_4. \end{aligned}$$

This shows that the linear map  $A'_l : H \rightarrow \Gamma^1(B)$  defined by

$$A'_l(h) := (id \otimes \varepsilon) \circ D'_l$$

has the form

$$A'_l(h) = \sum \tau_{\alpha_l^{-1}}(h_1) A_l(h_2) \tau_{\alpha_l}(h_3) + \sum \tau_{\alpha_l^{-1}}(h_1) d \tau_{\alpha_l}(h_2). \quad (30)$$

The same calculation for right connections leads to

$$A'_r(h) = \sum \tau_{\alpha_r}(h_3) A_r(h_2) \tau_{\alpha_r^{-1}}(h_1) - \sum \tau_{\alpha_r}(h_2) d \tau_{\alpha_r^{-1}}(h_1). \quad (31)$$

In general the linear maps  $A'_{l,r}$  do not satisfy the conditions (79) and (80) of [4] respectively, i.e.  $D'_{l,r}$  do in general not define a connection. Only in the special case when  $\Gamma(H)$  is the universal differential algebra, which means every linear map  $A : H \rightarrow \Omega^1(B)$  satisfying  $A(1) = 0$  defines a left and a right connection, every gauge transformation transforms connections in connections. Because of this problem it seems to be necessary to introduce the following definition.

**Definition 4** Let  $\mathcal{P}$  be locally trivial QPFB and let  $\alpha_{l,r}$  be a left(right) gauge transformation respectively. A left connection defined by  $hor_l$  is called  $\alpha_l$ -covariant if  $D'_l$  defined by

$$D'_l := \alpha_l \circ hor_l \circ d \circ \alpha_l^{-1}$$

defines the left covariant derivation of a left connection.

A right connection defined by  $hor_r$  is called  $\alpha_r$ -covariant if  $D'_r$  defined by

$$D'_r := \alpha_r \circ hor_r \circ d \circ \alpha_r^{-1}$$

defines the right covariant derivation of a right connection.

A left (right) connection is called covariant, if it is  $\alpha_{l,r}$ -covariant for all gauge transformations  $\alpha_{l,r}$ .

**Definition 5** Let  $\mathcal{G}_{l,r} \subset G_{l,r}$  be a subgroup. A left (right) connection is  $\mathcal{G}_{l,r}$ -covariant if it is  $\alpha_{l,r}$ -covariant for all  $\alpha_{l,r} \in \mathcal{G}_{l,r}$ .

**Proposition 10** Let  $\mathcal{P}$  be a locally trivial QPFB and let  $\Gamma(\mathcal{P})$  be a differential structure of  $\mathcal{P}$  where the differential algebra  $\Gamma(H)$  is the universal one. All left (right) connections are covariant.

For the proof see the remarks above.

**Proposition 11** Let  $\mathcal{P}$  be a locally trivial QPFB and let  $\Gamma(\mathcal{P})$  be a differential structure on  $\mathcal{P}$  where  $\Gamma(H)$  is bicovariant, i.e. the corresponding right ideal  $R$  is Ad-invariant. Let  $\mathcal{Q} \subset G_{l,r}$  be the subgroup of all gauge transformations which are differentiable algebra isomorphisms. All left and right connections are  $\mathcal{Q}$ -covariant.

Proof: It is sufficient to prove this assertion on a trivial bundle  $B \otimes H$ . As noted at the beginning of Section 1, an algebra automorphism  $\alpha : B \otimes H \rightarrow B \otimes H$  which is a gauge transformation corresponds to a homomorphism  $\tau_\alpha : H \rightarrow B$  with the property that  $\tau_\alpha(H)$  lies in the center of  $B$ . We have assumed that  $\alpha$  is differentiable with respect to  $\Gamma(B) \hat{\otimes} \Gamma(H)$ . Let  $J(B \otimes H) \subset \Omega(B \otimes H)$  be the differential ideal corresponding to  $\Gamma(B) \hat{\otimes} \Gamma(H)$ . The assumption that  $\alpha$  is differentiable means  $\alpha_\Omega(J(B \otimes H)) = J(B \otimes H)$ . As shown in [4], Proposition 4,  $J(B \otimes H)$  is generated by the sets

$$(id \otimes 1)_\Omega(J(B)); \{ \sum (1 \otimes S^{-1}(r_2))d(1 \otimes r_1) \mid r \in R \}; \\ \{ (a \otimes 1)d(1 \otimes h) - (d(1 \otimes h))(a \otimes 1) \mid a \in B, h \in H \}.$$

Applying  $\alpha_\Omega$  to these sets and using the Ad-invariance of  $R$  gives the following identities in  $\Gamma(B)$ .

$$\sum \tau_{\alpha^{-1}}(r_1)d\tau_\alpha(r_2) = 0, \forall r \in R \\ (da)\tau_\alpha(h) = \tau_\alpha(h)da, \forall a \in B, \forall h \in H.$$

Let  $A_{l,r}$  be the linear map corresponding to a connection on  $B \otimes H$ . Using formula (30) one obtains for the transformed linear maps  $A'_{l,r}$

$$A'_{l,r}(h) = \sum A_{l,r}(h_2)\tau_\alpha(S(h_1)h_3) + \sum \tau_{\alpha^{-1}}(h_1)d\tau_\alpha(h_2).$$

Inserting an element  $r \in R$  in this equation and using the Ad-invariance of  $R$ , i.e.  $\sum r_2 \otimes S(r_1)r_3 \in R \otimes H$ , and the properties of  $\tau_\alpha$  one obtains

$$A'_{l,r}(r) = 0, \forall r \in R.$$

Thus,  $A'_{l,r}$  defines a left (right) connection again. □

**Proposition 12** Let  $\mathcal{P}$  be a locally trivial QPFB. Assume there is given a differential structure on  $\mathcal{P}$ . Let  $D_{l,r}$  be a left (right) covariant derivative and  $\alpha_{l,r}$  a left (right) gauge transformation. The Map  $D'_{l,r}$  defined by

$$D'_{l,r} := \alpha_{l,r} \circ D_{l,r} \circ \alpha_{l,r}^{-1}$$

is a left (right) covariant derivative.

We leave the proof for the reader.

**Proposition 13** *Let  $\mathcal{P}$  be a locally trivial QPFB, let  $D_{l,r}$  be a covariant derivative and let  $\alpha_{l,r}$  be a gauge transformation.*

*Then there are the following transformation formulas for the left (right) curvature form  $\Omega_{l,r}$ :*

$$\Omega'_l(h) = \sum g_{\alpha_l^{-1}}(h_1)\Omega(h_2)g_{\alpha_l}(h_3) \quad (32)$$

$$\Omega'_r(h) = \sum g_{\alpha_r}(h_3)\Omega(h_2)g_{\alpha_r^{-1}}(h_1) \quad (33)$$

Proof: Using formulas (105) and (106) of [4] and (24) and (25) one obtains

$$(D'_{l,r})^2 = \alpha_{l,r} \circ (D_{l,r})^2 \alpha_{l,r}^{-1}$$

and

$$\begin{aligned} \alpha_l \circ (D_l)^2 \circ \alpha_l^{-1}(f) &= \sum f_0 g_{\alpha_l^{-1}}(f_1) \Omega_l(f_2) g_{\alpha_l}(f_3) \\ \alpha_r \circ (D_r)^2 \circ \alpha_r^{-1}(f) &= \sum g_{\alpha_r}(f_3) \Omega_r(f_2) g_{\alpha_r^{-1}}(f_1) f_0. \end{aligned}$$

for  $f \in \mathcal{P}$ . □

At the end of this section let us remark that on locally trivial QVB every gauge transformation  $\eta$  transforms connections  $\nabla$  in connections  $\nabla'$  by

$$\nabla' = \eta \circ \nabla \circ \eta^{-1}.$$

This is analogous to Proposition 12.

### 3 Example

This example is constructed to show that there exist nonclassical gauge transformations. Here we glue together a noncommutative “tube” with the quantum group  $SU_\nu(2)$  along the classical subspace  $S^1$  and construct a  $SU_\nu(2)$  bundle over this “base”. First, we define the algebra over the noncommutative “tube” as the algebra  $B_1$  generated by the elements

$$x, x^*, y = y^*$$

satisfying the relations

$$\begin{aligned} xx^* &= x^*x = 1 \\ xy &= qyx \\ x^*y &= q^{-1}yx^*, \end{aligned}$$

where  $q \in (0, 1]$ . It is easy to see that there exists a surjective homomorphism  $\pi_2^1 : B \longrightarrow P(S^1)$  defined by

$$\begin{aligned} \pi_2^1(x) &= a \\ \pi_2^1(x^*) &= a^* \\ \pi_2^1(y) &= 0, \end{aligned}$$

where  $a$  is the generator of  $P(U(1))$ .

There also exists a surjective homomorphism  $\pi_1^2 : P(SU_\nu(2)) \longrightarrow P(S^1)$  defined by

$$\begin{aligned} \pi_1^2(\alpha) &= a \\ \pi_1^2(\alpha^*) &= a^* \\ \pi_1^2(\gamma) &= \pi_1^2(\gamma^*) = 0, \end{aligned}$$

where  $\alpha, \gamma$  are the usual generators of  $P(SU_\nu(2))$ . Our basis algebra  $B$  is defined as the gluing of  $B_1$  and  $P(SU_\nu(2))$  by means of  $\pi_2^1$  and  $\pi_1^2$ ,

$$B := \{(f_1, f_2) \in B_1 \bigoplus P(SU_\nu(2)) \mid \pi_2^1(f_1) = \pi_1^2(f_2)\}.$$

Now one chooses the transition functions  $\tau_{ij} : P(SU_\nu(2)) \longrightarrow P(S^1)$  as follows:

$$\begin{aligned} \tau_{12}(\alpha) &= a \\ \tau_{12}(\alpha^*) &= a^* \\ \tau_{12}(\gamma) &= \tau_{12}(\gamma^*) = 0 \end{aligned}$$

and obtains a locally trivial QPFB  $\mathcal{P}$  with “structure group”  $SU_\nu(2)$ .

According to Proposition 4, to construct a left gauge transformation  $\alpha_l$  we have to find a linear map  $g_{\alpha_l} : P(SU_\nu(2)) \longrightarrow \mathcal{P}$  fulfilling (10) - (14).

First we define the linear maps  $\tau_1^{(n)} : P(SU_\nu(2)) \longrightarrow B_1$  and  $\tau_2^{(n)} : P(SU_\nu(2)) \longrightarrow P(SU_\nu(2))$ ,  $n \geq 1$ .  $\tau_1^{(n)}$  is assumed to be a homomorphism defined by

$$\begin{aligned} \tau_1^{(n)}(\alpha) &= x^n \\ \tau_1^{(n)}(\alpha^*) &= x^{*n} \\ \tau_1^{(n)}(\gamma) &= \tau_1^{(n)}(\gamma^*) = 0. \end{aligned}$$

$\tau_2^{(n)}$  is defined as follows:

$$\tau_2^{(n)}(h) = \sum h_1 h_2 \dots h_n.$$

The linear maps  $\tau_i^{(n)}$  are convolution invertible with convolution inverse

$$\tau_1^{(n)-1}(h) = \tau_1^{(n)}(S(h))$$

and

$$\tau_2^{(n)-1}(h) = \sum S(h_1) S(h_2) \dots S(h_n).$$

By an easy calculation one obtains the identity

$$\pi_2^1(\tau_1^{(n)}(h)) = \sum \tau_{12}(h_1) \pi_1^2(\tau_2^{(n)}(h_2)) \tau_{21}(h_3) = \pi_1^2(\tau_2^{(n)}(h)),$$

hence (see formula (22)) there exists a convolution invertible map  $g_{\alpha_l}^{(n)} : P(SU_\nu(2)) \longrightarrow \mathcal{P}$  which determines a left gauge transformation.

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